REAL ANALYSIS HOMEWORK 5

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1. Problem 1

Firstly, we have that $(a,\infty)^c = (-\infty, a]$. Using this, $(-\infty, a) = \bigcup_{n=1}^{\infty}(-\infty, a - 1/n]$. Also, $(-\infty, a)^c = [a, \infty)$. But now, since this shows the sets $(-\infty, a)$, $(-\infty, a]$, and $[a, \infty)$ can be constructed from the set (a, ∞), we have:

$$(a, b) = (-\infty, b) \cap (a, \infty)$$

$$[a, b) = (-\infty, b) \cap [a, \infty)$$

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Can also be constructed from the sets of the form (a, ∞), and we are done.

2. Problem 2

(a). Take $F_i := E_1 \backslash E_i$, and define $E := \lim\limits_{n\to\infty} E_n$. Then F_i is now an increasing sequence, and $F_i \to E_1 \backslash E$. However, for an increasing sequence, we have that $\lim\limits_{n\to\infty} m(F_n) = m(\lim\limits_{n\to\infty} F_n)$. Also, since $E_n \subset E_1$, $m(F_n) = m(E_1) - m(E_n)$, and we have the following:

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$$m(E_1) - m(E) = m(E_1 \backslash E) = \lim_{n\to\infty} (E_1 \backslash E_n) = \lim_{n\to\infty} (m(E_1) - m(E_n))$$

And we see that $m(E_1) - m(E) = m(E_1) - \lim_{n\to\infty} E_n$. Since $m(E_1) < \infty$, we can subtract $m(E_1)$ to conclude that $m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

(b). Define $E_n := [n, \infty)$. Then, this is obviously a decreasing sequence. Note that $E_n \to \varnothing$. To see this, just notice that if $a \in \mathbb{R}$, then find any integer N such that $N > a$. Then $a \notin E_N$.

It is easy to see that $m(E_n) = \infty$ for all n, and hence $\lim_{n\to\infty} m(E_n) = \infty$. However, by the above, $m(\lim_{n\to\infty} E_n) = m(\varnothing) = 0$. These are clearly not equal, and we are done.

3. Problem 3

By definition, $\liminf_{n\to\infty} = \lim_{n\to\infty} \bigcap_{k\geq n} E_k$. Define $F_n := \bigcap_{k\geq n} E_k$. Then, F_n is an increasing sequence, as increasing n removes sets from the intersection, which can only increase its size. Hence,

$$m(\lim_{n\to\infty} F_n) = \lim_{n\to\infty} m(F_n)$$

However, consider F_n. It is contained in every E_k for $k \geq n$. In other words, $m(F_n) \leq m(E_k)$ for $k \geq n$. Taking the infimum over all $k \geq n$,

$$m(F_n) \leq \inf_{k\geq n} m(E_k)$$

And, combining this with the above, we find:

$$m(\lim_{n\to\infty} F_n) \leq \lim_{n\to\infty} \inf_{k\geq n} m(E_k)$$

But by definition, this is precisely the statement $m(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} m(E_n)$.

4. Problem 4

Set $F_n := \bigcup_{k\geq n} E_k$. Then, F_n is a decreasing sequence of sets, and note that $m(F_1) \leq \sum_{n\geq 1} m(E_n) < \infty$, so we can apply the result of Problem 2 (a) to see:

$$m(\lim_{n\to\infty} F_n) = \lim_{n\to\infty} m(F_n) \leq \lim_{n\to\infty} \sum_{k\geq n} m(E_k)$$

However, since the series $\sum_{n\geq 1} m(E_n) < \infty$, we know that $\sum_{k\geq n} m(E_k) \to 0$ as $n \to \infty$ (this has to happen since the terms are nonnegative). Hence, noting that $\lim_{n\to\infty} F_n = \limsup_{n\to\infty} E_n$, we find that $m(\limsup_{n\to\infty} E_n) \leq 0$, so in fact $m(\limsup_{n\to\infty} E_n) = 0$, and we are done.

5. Problem 5

Assume for sake of contradiction that E is measurable and that $b > a$. Then, choose $0 < \epsilon < b - a$. There exists an open set $G \supset E$ and $F \subset E$ such that $m(G) - m(E) < \epsilon/2$ and $m(E) - m(F) < \epsilon/2$. Adding these, we find:

$$m(G) - m(F) < \epsilon < b - a$$

But this is clearly impossible by the definition of a and b, hence E cannot be measurable.